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Topological Electric Charge

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By treating magnetic charge as a gauge symmetry through the introduction of a "magnetic" pseudo four-vector potential, we show that it is possible to construct a topological electric charge from a theory which originally contains gauge magnetic charge. This is an explicit realization of the Montonen–Olive conjecture that there should exist a dual theory to the usual 't Hooft–Polyakov monopole theory in which the roles of the gauge and topological charges are reversed. The physical distinction between 't Hooft–Polyakov monopoles and the dual theory with electric charge is that the strong and weak coupling regimes are reversed. Physically this leads to the mass of the electrically charged soliton being on the order of $(1/137)M_W$ as opposed to the much larger mass (on the order of $137M_W$) of the magnetically charged soliton. Thus even for M_W in the TeV range such an electrically charged particle could be observed at some future accelerator.

1. INTRODUCTION

Several years ago 't Hooft (1974) and Polyakov (1974) showed how to construct a magnetically charged object starting from a theory with a non-Abelian gauge field coupled to a Higgs field. The gauge coupling of the non-Abelian theory was set equal to the electromagnetic coupling constant e, so that one could identify the $U(1)$ symmetry which remained after spontaneous symmetry breaking (SSB) with the usual electromagnetic gauge symmetry. A magnetically charged object was then generated by giving the scalar field an unusual topological vacuum structure at spatial infinity. This strange vacuum configuration was called the "hedgehog" solution by Polyakov since the direction in isospin space in which the vacuum expectation value (VEV) points is linked to the radial direction of ordinary space. This magnetic soliton, which emerges from a theory which originally has only electric gauge charge, has several unique properties. First, its magnetic charge is not a Noether charge, but a topological charge, which owes its existence to the

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unusual vacuum of the scalar field (Arafune *et al.*, 1975). Second, the monopole has no singularities in its fields. Finally, and unfortunately, it is estimated to have a mass on the order of $137 M_W$ (where M_W is the mass of the gauge bosons after symmetry breaking). If M_W is on the order of the electroweak gauge bosons (~ 100 GeV), then seeing such magnetically charged objects is out of the question for any current or planned accelerator. The monopole's large mass comes about because of the small value of the non-Abelian gauge coupling that one must choose in order that the embedded $U(1)$ symmetry can be identified with the usual Abelian gauge group of electromagnetism. This also leads to the monopole having a large magnetic charge of $4\pi/e$.

It might be asked if it is possible to construct such topologically stable solitons which have the far field of an electric charge. Julia and Zee (1975) found field configurations which carry both magnetic and electric charge, which are called dyons. However, for the dyon, the electric charge cannot exist without an accompanying magnetic charge, and the stability arguments that apply to the purely magnetic solution do not apply to the dyonic solution (although there are plausibility arguments for its stability). Purely electrically charged solitons, with a "small" mass, would be of interest phenomenologically, since one might be able to identify such objects with observed or observable particles. Such electrically charged solitons would not suffer from the singularity of the Coulomb field at the origin, which is found in classical point particles such as the electron. Rather, the Coulomb potential would evolve smoothly into a non-Coulomb, nonsingular field at the origin. Additionally, since we will find that the mass of these electric solitons is $\approx (1/137) M_W$, they would be found before the massive gauge bosons and so would provide a window on the non-Abelian gauge group. This is the opposite to 't Hooft–Polyakov monopoles, which have a larger mass than the massive gauge bosons of the theory.

The reason for thinking that electric solitons are possible is the dual symmetry (Jackson, 1975) between electric and magnetic quantities of Maxwell's equations

$$\begin{aligned} \mathbf{E} &\rightarrow \cos\theta\, \mathbf{E} + \sin\theta\, \mathbf{B} \\ \mathbf{B} &\rightarrow -\sin\theta\, \mathbf{E} + \cos\theta\, \mathbf{B} \end{aligned} \tag{1}$$

and

$$\begin{aligned} \rho_e &\rightarrow \cos\theta\, \rho_e + \sin\theta\, \rho_m, \qquad & \mathbf{J}_e &\rightarrow \cos\theta\, \mathbf{J}_e + \sin\theta\, \mathbf{J}_m \\ \rho_m &\rightarrow -\sin\theta\, \rho_e + \cos\theta\, \rho_m, \qquad & \mathbf{J}_m &\rightarrow -\sin\theta\, \mathbf{J}_e + \cos\theta\, \mathbf{J}_m \end{aligned} \tag{2}$$

where $\rho_{e(m)}$ and $\mathbf{J}_{e(m)}$ are the electric (magnetic) charge and current densities. Given a particle with a certain electric and magnetic charge, it is possible to use this dual symmetry to "rotate" the two charges so that the particle ends up with a different electric and magnetic charge. By properly choosing the angle θ, a particle can be made to carry only electric charge or only magnetic charge. The ability to altogether transform away one type of charge holds only if all particles have the same ratio of electric to magnetic charge, since the dual transformation of equation (2) is global.

Using the dual symmetry between electric and magnetic quantities, it should be possible to find a topological electric soliton by applying a duality transformation on the magnetic soliton. This possibility was conjectured to occur some time ago by Montonen and Olive (1977). Based on the dual symmetry between electric and magnetic quantities given above, they argued that there should exist a theory which was dual to that of 't Hooft and Polyakov, such that instead of starting with an electric gauge charge and ending up with a topological magnetic charge, one could start with a magnetic gauge charge and end up with a topological electric charge. In addition to switching the roles of the electric and magnetic quantities in the theory, this would change the strong- and weak-coupling regimes. In the case of the magnetic soliton one begins with a small electric charge and ends up with an enormous magnetic charge. For the electric soliton one expects the initial gauge charge to be large, while the final electric charge is small. We shall see that this does happen, with the physical consequence that the mass of the electric soliton is relatively small (i.e., small enough that it would be feasible to observe such an electrically charged object at some future accelerator).

The main obstacle to carrying through the Montonen–Olive conjecture is that the 't Hooft–Polyakov construction occurs at the level of the gauge potentials. When **E** and **B** are written in terms of the four-vector potential A_μ ($E_i = \partial^i A^0 - \partial^0 A^i$ and $B_i = -\partial^j A^k + \partial^k A^j$), it appears impossible to implement the dual symmetry of equation (1) in terms of the potentials. What is needed is a formulation of electromagnetism that is symmetric at the level of the gauge potentials. Various authors (Cabibbo and Ferrari, 1962; Rohrlich, 1966; Zwanziger, 1971) have accomplished this by introducing a second, pseudo four-vector potential C_μ in addition to the usual four-vector potential A_μ (the term pseudo for C_μ refers to its behavior under parity). This two-potential approach has the advantage over the Dirac string approach (Dirac, 1931; 1948) or the Wu–Yang fiber bundle approach (Wu and Yang, 1975) in that it requires neither a singular string variable nor a patching of the gauge potential. Using two potentials also puts magnetic charge on the same footing as electric charge by treating both as $U(1)$ gauge symmetries (Carmeli, 1982). The drawback of this approach is that there are two "photons" in the theory rather than the one photon that is observed (Hagen, 1965). This can be overcome in two ways: Either by putting extra conditions on the two gauge fields so that only the number of degrees of freedom necessary for one photon is left (Zwanziger, 1971); or by accepting the other "photon," but hiding it and the magnetic charge associated with it through the Higgs mechanism (Singleton, 1995).

The two-potential theory of electric and magnetic charge allows the dual symmetry of Maxwell's equations to be extended to the level of the gauge fields. Using this with the 't Hooft–Polyakov construction, one easily constructs topologically stable electric poles rather than magnetic poles. The major difference between the magnetically charged soliton and the electrically charged soliton is in the enormous difference of their masses. We will first review the relevant aspects of the two-potential theory.

2. THE DUAL FOUR-VECTOR POTENTIAL

In three-vector notation, Maxwell's equations with electric and magnetic charge are (in Lorentz–Heaviside units) (Jackson, 1975)

$$\nabla \cdot \mathbf{E} = \rho_e, \qquad \nabla \times \mathbf{B} = \frac{1}{c}\left(\frac{\partial \mathbf{E}}{\partial t} + \mathbf{J}_e\right)$$

$$\nabla \cdot \mathbf{B} = \rho_m, \qquad -\nabla \times \mathbf{E} = \frac{1}{c}\left(\frac{\partial \mathbf{B}}{\partial t} + \mathbf{J}_m\right) \tag{3}$$

Introducing two four-vector potentials $A^\mu = (\phi_e, \mathbf{A})$ and $C^\mu = (\phi_m, \mathbf{C})$, we can write the $\mathbf{E}$ and $\mathbf{B}$ fields as

$$\mathbf{E} = -\nabla\phi_e - \frac{1}{c}\frac{\partial \mathbf{A}}{\partial t} - \nabla \times \mathbf{C}$$

$$\mathbf{B} = -\nabla\phi_m - \frac{1}{c}\frac{\partial \mathbf{C}}{\partial t} + \nabla \times \mathbf{A} \tag{4}$$

The usual definitions of $\mathbf{E}$ and $\mathbf{B}$ only involve ϕ_e and $\mathbf{A}$. Substituting the above expanded definitions for $\mathbf{E}$ and $\mathbf{B}$ into Maxwell's equations (3) yields (after using some standard vector identities and applying the Lorentz gauge condition to both four-vector potentials) the wave equation form of Maxwell's equations for both A^μ and C^μ. The equation for A^μ has electric charges and currents $[J_e^\mu \equiv (\rho_e, \mathbf{J}_e)]$ as sources, while the equation for C^μ has magnetic charges and currents $[J_m^\mu \equiv (\rho_m, \mathbf{J}_m)]$ as sources. In the two-potential theory all of Maxwell's equations are dynamical equations.

The two four-vector potentials A^μ and C^μ are similar except for their behavior under parity transformations. The $\mathbf{E}$ field is an ordinary vector under parity, and the $\mathbf{B}$ field is a pseudovector. The normal definition of the fields in terms of the potentials implies that ϕ_e must be a scalar and $\mathbf{A}$ must be a vector under parity. In order for the **E** and **B** fields to retain their parity properties under the expanded definitions of equations (4), ϕ_m must be a pseudoscalar and **C** must be a pseudovector under parity. Therefore A^μ is a four-vector, while C^μ is a pseudo four-vector.

The two-potential theory can be cast most simply in four-vector notation. We define the two field strength tensors (where $c = 1$ from now on)

$$
\begin{aligned}
F^{\mu\nu} &= \partial^\mu A^\nu - \partial^\nu A^\mu \\
G^{\mu\nu} &= \partial^\mu C^\nu - \partial^\nu C^\mu
\end{aligned}
\tag{5}
$$

and their duals

$$
\begin{aligned}
\mathcal{F}^{\mu\nu} &= \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}F_{\alpha\beta} \\
\mathcal{G}^{\mu\nu} &= \frac{1}{2}\,\epsilon^{\mu\nu\alpha\beta}G_{\alpha\beta}
\end{aligned}
\tag{6}
$$

where $\epsilon^{\mu\nu\alpha\beta}$ is the Levi-Civita tensor, with $\epsilon^{0123} = +1$, and having total antisymmetry in its indices. The **E** and **B** fields can then be written as

$$
\begin{aligned}
E_i &= F^{i0} - \mathcal{G}^{i0} = F^{i0} + \frac{1}{2}\,\epsilon^{ijk}G_{jk} \\
B_i &= G^{i0} + \mathcal{F}^{i0} = G^{i0} - \frac{1}{2}\,\epsilon^{ijk}F_{jk}
\end{aligned}
\tag{7}
$$

It is the $\mathcal{G}^{i0}$ part of the **E** field which is crucial, since it is this term which gives rise to the electric, Coulomb far-field of the soliton when the magnetic $U(1)$ gauge symmetry is embedded into the non-Abelian theory via 't Hooft's generalized electromagnetic field strength tensor. This $\mathcal{G}^{i0}$ term is absent in the usual formulation of electromagnetism. Maxwell's equations in four-vector notation become

$$
\begin{aligned}
\partial_\mu F^{\mu\nu} &= \partial_\mu \partial^\mu A^\nu = J_e^\nu \\
\partial_\mu G^{\mu\nu} &= \partial_\mu \partial^\mu C^\nu = J_m^\nu
\end{aligned}
\tag{8}
$$

where the Lorentz condition ($\partial_\mu A^\mu = \partial_\mu C^\mu = 0$) has been imposed on the two potentials in going from the first to the middle expression. Finally, the dual symmetry of equations (1), (2) can now be written in terms of the two four-vector potentials and the two four-currents

$$
\begin{aligned}
A^\mu &\to \cos\theta\, A^\mu + \sin\theta\, C^\mu, \qquad & C^\mu &\to -\sin\theta\, A^\mu + \cos\theta\, C^\mu \\
J_e^\mu &\to \cos\theta\, J_e^\mu + \sin\theta\, J_m^\mu, \qquad & J_m^\mu &\to -\sin\theta\, J_e^\mu + \cos\theta\, J_m^\mu
\end{aligned}
\tag{9}
$$

Equation (9) extends the dual symmetry of equation (1) to the level of the four-vector potentials. This implies that it should be possible to construct a topological electric charge in exactly the same way 't Hooft and Polyakov constructed a topological magnetic charge. The relevant parts of their solution are reviewed in the next section.

3. THE 'T HOOFT–POLYAKOV MONOPOLE SOLUTION

't Hooft (1974) and Polyakov (1974) independently discovered the possibility of constructing a finite-energy, magnetically charged soliton in a non-Abelian gauge theory coupled to a Higgs field. The stability of this field configuration was guaranteed by the nontrivial homotopy of the Higgs field (Arafune *et al.*, 1975).

In constructing the monopole solution 't Hooft considered an $SO(3)$ gauge theory coupled to a triplet scalar field with the following Lagrangian density:

$$\mathcal{L} = -\frac{1}{4} H^a_{\mu\nu} H^{a\mu\nu} + \frac{1}{2} D_\mu \Phi^a D^\mu \Phi^a + \frac{1}{2} \mu^2 \Phi^a \Phi^a - \frac{1}{4} \lambda (\Phi^a \Phi^a)^2 \tag{10}$$

where

$$H^a_{\mu\nu} = \partial_\mu W^a_\nu - \partial_\nu W^a_\mu + g\epsilon^{abc} W^b_\mu W^c_\nu \tag{11}$$

and

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g\epsilon^{abc} W^b_\mu \Phi^c \tag{12}$$

ϵ^{abc} are the structure constants of $SO(3)$. The $SO(3)$ gauge coupling g is at this point unspecified. If $\mu^2 > 0$ and $\lambda > 0$, then the scalar field develops a vacuum expectation value of $v = \mu/\sqrt{\lambda}$, reducing the $SO(3)$ symmetry to a $U(1)$ symmetry. Inserting the spherically symmetric ansatz

$$W^a_i = \frac{\epsilon_{aij} x^j [1 - K(r)]}{gr^2}, \qquad W^a_0 = 0$$

$$\Phi^a = \frac{x^a H(r)}{gr^2} \tag{13}$$

into the equations of motion that come from the Lagrangian of equation (10), one arrives at two coupled differential equations for the functions $K(r)$ and $H(r)$ (Julia and Zee, 1975)

$$r^2 K'' = K(K^2 + H^2 - 1)$$

$$r^2 H'' = 2HK^2 + \frac{\lambda}{g^2} H(H^2 - g^2 v^2 r^2) \tag{14}$$

where the primes indicate differentiation with respect to r. These equations must be solved numerically, except in the special case when μ, $\lambda = 0$ (Prasad and Sommerfield, 1975), where an analytical solution can be found. In addition to the pure gauge solution to these equations [i.e., $K(r) = 0$ and $H(r) = (g\mu/\sqrt{\lambda})r$], there exists a nontrivial finite-energy solution. That such a solution exists can best be seen by calculating the energy of the field configuration of equation (13),

$$
\begin{aligned}
E &= \int T^{00}(r)\, d^3x \\
&= \int \left(\frac{1}{4} H^a_{ij}H^{aij} - \frac{1}{2} D_i\Phi^a D^i\Phi^a - \frac{1}{2}\mu^2\Phi^a\Phi^a + \frac{1}{4}\lambda(\Phi^a\Phi^a)^2\right) d^3x \\
&= \frac{4\pi}{g^2}\int_0^\infty \left((K')^2 + \frac{(K^2-1)^2}{2r^2} + \frac{H^2K^2}{r^2} + \frac{(rH' - H)^2}{2r^2}\right. \\
&\quad \left. + \frac{\lambda}{4} r^2\left(\frac{H^2}{g^2r^2} - v^2\right)^2\right) dr
\end{aligned} \tag{15}
$$

where the constant term $\frac{1}{4}\lambda v^4$ was added to the expression for the energy, so that the scalar potential term could be written as the square of some quantity. Thus every term in equation (15) is positive-definite. Then, since neither $K(r) = 0$ nor $K(r) = 1$ is the lowest minimum, the variational principle requires that an intermediate solution must exist. In the absence of a scalar field one finds that $K(r) = 0$ minimizes the energy. This is the Wu–Yang solution, which is singular at the origin. Thus the scalar field is crucial to obtaining a nonsingular, finite-energy solution. When μ^2 and λ are nonzero the solution must be found numerically, and then equation (15) becomes

$$E = \frac{4\pi}{g^2} M_W f\left(\frac{\lambda}{g^2}\right) \tag{16}$$

where $M_W = \mu g/\sqrt{\lambda}$ is the mass of two of the $SO(3)$ gauge bosons after symmetry breaking. The function $f(\lambda/g^2)$, which must be evaluated numerically, is $\mathcal{O}(1)$.

The Abelian $U(1)$ symmetry, which remains after the $SO(3)$ is spontaneously broken, can be identified with the usual electromagnetic gauge symmetry by making the following gauge-invariant generalization of the Maxwell field strength tensor:

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \frac{1}{g|\Phi|^3}\epsilon_{abc}\Phi^a(\partial_\mu\Phi^b)(\partial_\nu\Phi^c)$$

$$A_\mu = \frac{1}{|\Phi|}\Phi^a W^a_\mu \tag{17}$$

This definition reduces to the usual definition of the field strength tensor if one considers the unitary gauge where the scalar field is gauge rotated so that it points in only one direction in the internal $SO(3)$ space. This will be discussed more later. To see how a monopole emerges from this generalized field strength tensor, the asymptotic values of the ansatz of equation (13) are inserted into equation (17). As $r \to \infty$, $K(r) \to 0$ and $H(r) \to (g\mu/\sqrt{\lambda})r$, which means

$$W^a_i \to \epsilon_{aij}x^j/gr^2 + \mathcal{O}(r^{-2})$$

$$\Phi^a(r) \to x^a v/r + \mathcal{O}(r^{-2}) \tag{18}$$

The asymptotic configuration of the scalar field is called the "hedgehog" solution because of the peculiar way in which the Higgs field approaches its vacuum value v at spatial infinity. Instead of pointing in a fixed direction in isospin space for all points in configuration space (i.e., $\Phi^a(r) = v\delta^{a3} = v[0, 0, 1]$), it points in an isospin direction that coincides with the radial spatial direction. This links the internal (isospin) space with the external (configuration) space. Inserting the asymptotic fields of equation (18) into the generalized field strength tensor of equation (17) yields

$$B_i = -\frac{1}{2}\epsilon_{ijk}F^{jk} \to \frac{r^i}{gr^3}$$

$$E_i = F^{i0} = 0 \tag{19}$$

So as $r \to \infty$ the fields rapidly approach those of a magnetic Coulomb field and zero electric field. For the magnetic monopole one sets the non-Abelian gauge coupling equal to the usual electric $U(1)$ coupling ($g = e$). The magnetic charge implied by the far fields of equation (19) is then $4\pi/e$.

4. ELECTRICALLY CHARGED SOLITON

By setting $A_\mu = W^a_\mu\Phi^a/|\Phi|$, one identifies the $U(1)$ symmetry which remains after symmetry breaking with the usual "electric" Abelian gauge symmetry. However, using the dual four-vector potential formalism, one could just as easily set $C_\mu = W^a_\mu\Phi^a/|\Phi|$, so that the remaining $U(1)$ symmetry now corresponds to the "magnetic" Abelian gauge symmetry. The only change that this entails is that the $SO(3)$ gauge fields W_μ^a would have to be pseudo quantities under parity, since C_μ is a pseudo four-vector. (One could also consider having W_μ^a be regular under parity and let Φ^a be a pseudoscalar. This also makes C_μ a pseudo four-vector). This switching of the two four-vector potentials A_μ and C_μ can be seen as a dual rotation, with θ set to $\pi/2$ in equation (9). This gives $A_\mu \rightarrow C_\mu$, and from equation (5) it also changes the "electric" field strength tensor into the "magnetic" field strength tensor ($F_{\mu\nu} \rightarrow G_{\mu\nu}$). In this way equation (17) becomes

$$G_{\mu\nu} = \partial_\mu C_\nu - \partial_\nu C_\mu - \frac{1}{g|\Phi|^3} \epsilon_{abc} \Phi^a (\partial_\mu \Phi^b)(\partial_\nu \Phi^c)$$

$$C_\mu = \frac{1}{|\Phi|} \Phi^a W_\mu^a \tag{20}$$

As with the expression for the "electric" field strength tensor, $G_{\mu\nu}$ is gauge invariant. Inserting the asymptotic values of the non-Abelian gauge fields and of the scalar fields (which now define the magnetic gauge potential C_μ) from (18) into equation (20), one finds that the far fields of this soliton are

$$B_i = G^{i0} = 0$$

$$E_i = \frac{1}{2} \epsilon^{ijk} G_{jk} \rightarrow -\frac{r^i}{gr^3} \tag{21}$$

where the expanded definitions of the **E** and **B** fields from equation (7) have been used. Requiring that the charge of the **E** field in equation (21) have the magnitude of the charge of an electron (or proton) leads to the requirement that $g = 4\pi/e$ (where e is the magnitude of the electron's charge). This means that the original $SO(3)$ coupling g must be large. In the case of the magnetic soliton the non-Abelian gauge coupling was taken to be $g = e$, since there one wanted to embed the electric $U(1)$ symmetry into the $SO(3)$ theory. In the present case, the coupling of the magnetic $U(1)$ symmetry which is embedded in the $SO(3)$ theory is fixed by the condition that the electric charge of the soliton be that of observed particles. This is in accord with the Montonen–Olive conjecture that the weak- and strong-coupling regimes should exchange roles in the dual theory (i.e., in the 't Hooft–Polyakov case one had a weak electric gauge charge and ended up with an enormous topological magnetic charge. Here we start with a large magnetic gauge charge and end up with a small topological electric charge).

Up to this point all that has been accomplished can be viewed as simply a renaming of magnetic and electric quantities. However, there is a physical distinction between the two cases (i.e., $W_\mu^a \Phi^a / |\Phi|$ equaling either A_μ or C_μ). The difference lies in the classical masses of the two types of solitons. We will find that the mass of the electrically charged soliton is $\approx 10^{-4}$ times the mass of the magnetically charged soliton of 't Hooft and Polyakov. By equating the energy in the fields with the mass of the soliton one finds from equation (16), that the magnetically charged soliton (which has $g = e$) has a mass of

$$M_m = E = \frac{4\pi}{e^2} M_W f\left(\frac{\lambda}{e^2}\right) \approx 137 M_W \tag{22}$$

since numerically $f(\lambda/e^2)$ is $\mathbb{O}(1)$. In contrast the electric soliton (which has $g = 4\pi/e$) has a mass of

$$M_e = E = \frac{e^2}{4\pi} M_W f(\lambda e^2) \approx \frac{1}{137} M_W \tag{23}$$

The function $f(\lambda e^2)$ is $\mathbb{O}(1)$, since the energy of the fields [equation (15)] is invariant under the duality transformation that turns the magnetic soliton into an electric soliton. One might worry that the argument of f is different in the two cases. It can be shown that $f(0) = 1$ (Prasad and Sommerfield, 1975) and increases monotonically with the argument. Thus for a given λ the argument of the electric case is always closer to zero and the value of the function f is closer to 1. From equations (22), (23) it is seen that the mass of the electric soliton is over 10^4 times smaller than that of the magnetic soliton. If the mass of the gauge boson M_W is taken to be of the order of the electroweak gauge bosons [i.e., $\mathbb{O}(100)$ GeV], then such an electrically charged, spin-zero particle should have already been detected. This would seem to imply that if such electric solitons exist, the non-Abelian gauge group into which they are embedded must undergo symmetry breaking in such a way that the masses of the gauge bosons are at least several orders of magnitude greater than the masses of the electroweak gauge bosons. Even if M_W were in the range of 50 TeV it might be possible to see such an electric soliton at some reasonably extrapolated future accelerator. The observation of such an electrically charged soliton would precede the observation of the massive gauge bosons of the theory. In this way the soliton would provide a window on the higher energy scale of the spontaneously broken non-Abelian gauge group. An alternative possibility would be to use the spin from isospin mechanism (Jackiw and Rebbi, 1976; Hasenfratz and 't Hooft, 1976) and form bound states out of particles with various combinations of topological electric charge and gauge magnetic charge. These bound states would carry a spin of 1/2, obey Fermi–Dirac statistics (Goldhaber, 1976), and be in the mass range of the baryons. In this paper, however, we simply want to show the theoretical possibility of obtaining a topological electric charge from a non-Abelian gauge theory, since the $SO(3)$ group which is used here is apparently not a theory picked by nature. We will leave for a future work the task of building a more realistic model through the use of a larger non-Abelian symmetry.

In standard electrodynamics only the **B** field can be written as a curl. Including magnetic charge in electrodynamics as a gauge charge by introducing a second four-vector potential then requires that part of the **E** field be given by the curl of this second potential. The crucial element in constructing a finite-energy, stable field configuration with either a Coulomb electric or magnetic far field is being able to write that field as the curl of some vector potential. A general argument can be given (Cheng and Li, 1984) that shows this. In order for the energy of the soliton [equation (15)] to be finite, the covariant derivative of the scalar field must satisfy the following boundary condition as $r \to \infty$:

$$D_\mu \Phi^a = \partial_\mu \Phi^a + g\epsilon^{abc} W_\mu^b \Phi^c \to \mathcal{O}(r^{-2}) \tag{24}$$

In order for there to be a Coulomb far field (either electric or magnetic), the gauge fields W_μ^a must go like r^{-1} as $r \to \infty$. In addition, the magnitude of the scalar field must approach a constant (its VEV) as $r \to \infty$. Then both of the two separate terms in equation (24) need not approach zero like r^{-2}, since some cancellation can occur between the terms such that $D_\mu \Phi^a \to 0$ like r^{-2}. This is what happens with the ansatz (13). For time-independent fields the time component of the first term of equation (24) is zero, so no cancellation can occur between the two terms. Therefore W_0^a must go to zero faster than r^{-1} and does not give rise to a Coulomb far field. When the $U(1)$ gauge field is identified with $W_\mu^a \Phi^a / |\Phi|$, as in (17) or (20), this implies that the time component of the $U(1)$ gauge field also will not yield a Coulomb far field. In standard electrodynamics, where the **E** field is defined only by F^{i0}, a Coulomb field is only possible if $A_0 \neq 0$ (in fact if $A_0 = 0$ and only static solutions are considered then $\mathbf{E} = 0$). In the two-potential theory, however, the **E** field also has a part that is the curl of a vector potential (i.e., $E_i = 1/2\epsilon^{ijk} G_{jk}$). A Coulomb far field is then possible if the spatial components of the non-Abelian gauge field [and therefore the spatial components of the embedded $U(1)$ gauge field] go to zero like r^{-1}, as is the case for the ansatz (13).

Looking in detail at where the Coulomb far fields come from, it appears as if they are due entirely to the scalar fields. Inserting the asymptotic field conditions (18) into the generalized field strength tensors (17) or (20), it is found that the Coulomb fields come only from the last term of the generalized field strength tensors, which involve only the scalar fields. This makes it appear that whether the soliton has a magnetic charge or an electric charge is completely independent of the type of $U(1)$ gauge field (either A_μ or C_μ) that is embedded into the non-Abelian gauge theory. This is not the case. It has been shown (Arafune *et al.*, 1975) that by performing a gauge transformation to the unitary or Abelian gauge, it is possible to rotate the scalar field into the more common asymptotic vacuum configuration

$$\Phi^a(r) \to v\delta^{a3} = v(0, 0, 1) \tag{25}$$

Using this asymptotic scalar field in the generalized field strength tensors gives

$$\begin{aligned} G_{\mu\nu} &= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu \\ W^3_\mu &= C_\mu \end{aligned} \tag{26}$$

or

$$\begin{aligned} F_{\mu\nu} &= \partial_\mu W^3_\nu - \partial_\nu W^3_\mu \\ W^3_\mu &= A_\mu \end{aligned} \tag{27}$$

where the $U(1)$ gauge bosons are now associated exclusively with the third isospin component $SO(3)$ gauge boson. Since the expression for the generalized field strength tensors is gauge invariant, one still gets a Coulomb far field. This comes about even though the field strength tensors (26) and (27) are of the form that usually preclude the existence of a Coulomb field coming from the spatial part of the tensor, because the gauge transformation that takes the "hedgehog" gauge to the Abelian gauge is singular along the positive z-axis. The singularity in the gauge transformation gives rise to a similar singularity in W^3_μ. In this way a connection between the 't Hooft–Polyakov monopole and the Dirac monopole can be seen.

5. CONCLUSIONS

In this paper it has been shown that it is possible to construct a finite-energy, topologically stable soliton with electric charge. This is accomplished through the application of the 't Hooft–Polyakov monopole solution to the two-potential theory of electric and magnetic charge, where the two types of charges are treated as gauge charges through the introduction of two four-vector potentials. In the case of the magnetically charged soliton one starts with some non-Abelian gauge theory which is coupled to a scalar field that breaks the gauge symmetry. By embedding the electric $U(1)$ symmetry in the non-Abelian theory through the introduction of a generalized field strength tensor and taking the scalar field to go to the "hedgehog" solution as $r \to \infty$, it is found that a stable, finite-energy, magnetically charged soliton emerges. The gauge coupling g of the non-Abelian group is required to satisfy $g = e$ in order that the embedded $U(1)$ symmetry may be identified with the usual electric $U(1)$ gauge field. This leads to both the magnetic charge of the monopole ($=4\pi/e$) and the mass of the monopole ($\sim 4\pi/e^2$) being extremely large. Thus by starting with an electric gauge charge and using the 't Hooft–Polyakov ansatz, one ends up with a topological magnetic charge.

Using the dual symmetry between electric and magnetic quantities, we have shown that this construction can be reversed—i.e., start with a magnetic gauge charge and using the 't Hooft–Polyakov ansatz, end up with a topological electric charge. This is made possible by treating magnetic charge, like electric charge, as a gauge symmetry by introducing the pseudo four-vector potential C_μ. Using the dual transformation in terms of the potentials to "rotate" the electric potential A_μ into the magnetic potential C_μ, it is found that the magnetic soliton is transformed into an electric soliton. Requiring that the electric charge of this soliton be equal in magnitude to the charge of other electrically charged particles (e.g., electrons, protons), we found that the original non-Abelian gauge coupling must satisfy $g = 4\pi/e$. This made the mass of the electrically charged soliton several orders of magnitude lighter than its magnetic counterpart [$M_e \sim (1/137)M_W$ compared to $M_m \sim 137\, M_W$]. This construction of a topological electric charge with the properties found in this paper is an explicit realization of the Montonen–Olive conjecture. Taking M_W to be of the order of the electroweak gauge boson masses, such an electrically charged, spin-zero particle should have been observed. This might be taken to imply that such electric solitons are only of theoretical interest. However, by using the spin from isospin mechanism (Jackiw and Rebbi, 1976; Hasenfrantz and 't Hooft, 1976), it may be possible to construct bound states of topological electric charge and gauge magnetic charge which behave like spin-1/2 fermions (Goldhaber, 1976) and have masses roughly in the range of the baryon masses. Here, however, our goal was simply to show that it is possible to get a topological electric charge from a non-Abelian gauge theory, since the $SO(3)$ group is currently not thought to play a fundamental role in particle physics.

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